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ON COBB-DOUGLAS PREFERENCES IN BILATERAL OLIGOPOLY

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Abstract

Bilateral oligopoly is a simple model of exchange in which a finite set of sellers seek to exchange the goods they are endowed with for money with a finite set of buyers, and no price-taking assumptions are imposed. If trade takes place via a strategic market game bilateral oligopoly can be thought of as two linked proportional-sharing contests: in one the sellers share the aggregate bid from the buyers in proportion to their supply and in the other the buyers share the aggregate supply in proportion to their bids. The analysis can be separated into two ‘partial games’. First, fix the aggregate bid at B ; in the first partial game the sellers contest this fixed prize in proportion to their supply and the aggregate supply in the equilibrium of this game is $\tilde{X}(B)$. Next, fix the aggregate supply at X ; in the second partial game the buyers contest this fixed prize in proportion to their bids and the aggregate bid in the equilibrium of this game is $\tilde{B}(X)$. The analysis of these two partial games takes into account competition *within* each side of the market. Equilibrium in bilateral oligopoly must take into account competition *between* sellers and buyers and requires, for example, $\tilde{B}(\tilde{X}(B)) = B$. When all traders have Cobb-Douglas preferences $\tilde{X}(B)$ does not depend on B and $\tilde{B}(X)$ does not depend on X : whilst there is competition within each side of the market there is no strategic interdependence *between* the sides of the market. The Cobb-Douglas assumption provides a tractable framework in which to explore the features of fully strategic trade but it misses perhaps the most interesting feature of bilateral oligopoly, the implications of which are investigated.

Key words: strategic market game; bilateral oligopoly; Cobb-Douglas preferences; aggregative games.

JEL classification: C72; D43; D50.

1 Introduction

Strategic market games, originally introduced to the mainstream literature by Shapley and Shubik (1977) and since studied extensively (see Giraud (2003) for a review),

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model exchange between agents in a pure exchange economy without imposing any price-taking assumptions. These models are useful in two main respects: they allow investigation of the properties of equilibria when all traders are large and trade is ‘fully strategic’; and they allow consideration of whether traditional market models that impose price-taking assumptions, such as those of Cournot and Walras, have a strategic foundation. The first line of inquiry includes issues such as the existence and uniqueness of equilibrium; a comparison of equilibrium with that where it is assumed all or a subset of traders are price-takers; and study of the comparative static properties of the model. The second involves investigation of the sequence of equilibria as the number of traders who are assumed to be price takers in the model for which a strategic foundation is sought are increased without bound, to consider whether the sequence converges to the equilibrium of the model that imposes the price-taking assumption.

These are by no means trivial issues to address. When an issue is complex there is substantial value in exploring the model using an example that imposes some structure on the primitives. In doing so, caution must be exercised as the structure imposed by the example may imply some special features that are not true in general. A ‘good’ example will provide a tractable framework, but retain the main features of the general model. Assuming a ‘Cobb-Douglas economy’ (i.e. that all traders preferences are Cobb-Douglas) permits a highly tractable investigation of many issues in strategic market games. The Cobb-Douglas assumption has been used to good effect in the investigation of certain issues. For example, under the assumption of Cobb-Douglas preferences for one set of traders Codognato and Julien (2013) are able to extend the existence theorem of Busetto et. al. (2008) by weakening the assumptions required for the other set of traders. However, in bilateral oligopoly the assumption implies some very special features of the strategic market game that are specific to Cobb-Douglas preferences and do not hold in general. An analysis of fully strategic trade using a Cobb-Douglas economy can thus lead to spurious conclusions and does not give a clear picture of the nature of equilibrium. In a bilateral oligopoly framework, this paper investigates the special features that the Cobb-Douglas assumption imposes; investigates the implications for an analysis of strategic market games in this framework; and proposes an alternative example that retains most of the tractability of the Cobb-Douglas assumption but avoids its pitfalls.

Bilateral oligopoly (originally introduced by Gabszewicz and Michel (1997)) is perhaps the simplest economic environment in which a strategic market game mechanism can be used to model fully strategic trade, and has the flavor of a partial equilibrium framework. There are two commodities, interpreted as a standard consumption commodity and a commodity money, and two distinct sets of traders: sellers are endowed only with the good and buyers are endowed only with money. There is a trading post to which sellers may offer a portion of their endowment of the good to be exchanged for money, and buyers may bid a portion of their money to be exchanged for the good.

The trading post aggregates these offers and bids and allocates the aggregate amount of money bid amongst the sellers in proportion to their offers, and allocates the aggregate supply amongst the buyers in proportion to their bids. Bilateral oligopoly can thus be viewed as two linked (Tullock-style) contests in which the traders on each side of the market contest, in proportion to their actions, a ‘prize’ that is determined by the aggregate actions of the traders on the opposite side of the market.

Dickson and Hartley (2008) recognized that bilateral oligopoly can be analyzed as two ‘partial games’ that must be consistent with each other. First, fix the aggregate bid at B ; in the first partial game the sellers contest a fixed prize of B in proportion to their supply and the aggregate supply in the equilibrium of this game is $\tilde{X}(B)$. Next, fix the aggregate supply at X ; in the second partial game the buyers contest a fixed prize of X in proportion to their bids and the aggregate bid in the equilibrium of this game is $\tilde{B}(X)$. The analysis of these two partial games takes into account competition amongst traders *within* each side of the market. Equilibrium in bilateral oligopoly requires consistency between these partial games, for example for B to be such that $\tilde{B}(\tilde{X}(B)) = B$ so the equilibrium aggregate supply in the contest amongst the sellers in which the prize is B is such that, when this is contested by the buyers, their equilibrium aggregate bid is precisely B . Consistency between the partial games captures competition *between* the sides of the market.

It is particularly the strategic interdependence *between* the sides of the market that makes bilateral oligopoly a rich and interesting model: to deduce their optimal supply sellers have to form conjectures about the behaviour of buyers and buyers have to do the same for sellers, as well as consider the actions of their fellow competitors as in standard models. However, when it is assumed that all traders’ preferences are Cobb-Douglas there is no strategic interdependence between the sides of the market in bilateral oligopoly: for any $B, B' > 0$ $\tilde{X}(B) = \tilde{X}(B')$, and for any $X, X' > 0$ $\tilde{B}(X) = \tilde{B}(X')$. Whilst the Cobb-Douglas economy assumption allows for a tractable analysis of equilibrium it misses a key feature of the strategic interaction. It is shown that if an analysis of bilateral oligopoly is undertaken assuming a Cobb-Douglas economy it would be easy to conclude that whenever gains from trade exist there is an equilibrium with trade; that the effects of entry to the supply side concord with the conventional wisdom for Cournot markets that increased competition reduces sellers’ payoffs; and that as the number of buyers in bilateral oligopoly increases without bound the sequence of bilateral oligopoly equilibria converge to the Cournot equilibrium. These conclusions, however, are implied by the lack of strategic interdependence between the sides of the market and are not a general features of bilateral oligopoly.

Caution, therefore, is recommended to be exercised if the Cobb-Douglas economy assumption is invoked to gain tractability. This paper outlines the known pitfalls, but when investigating new research ideas an investigation of the model using the Cobb-Douglas assumption might miss key features due to the lack of strategic interdependence between the sides of the market. An alternative assumption that retains

most of the tractability of Cobb-Douglas preferences is that of quadratic preferences (that underlie linear competitive supply and demand functions), which also have the desirable feature that the strategic interdependency between sellers and buyers is preserved. To illustrate the tractability of this assumption an example is presented.

The rest of the paper is structured as follows. Section 2 sets out the model of bilateral oligopoly and Section 3 outlines the ‘dual contest’ nature of the game, discusses equilibria in the ‘partial games’ and equilibrium in bilateral oligopoly. Section 4 presents the relevant properties of Cobb-Douglas preferences, and Section 5 outlines the special features of the dual contest under the Cobb-Douglas assumption. Section 6 goes on to consider the known ‘pitfalls’ of assuming a Cobb-Douglas economy in bilateral oligopoly. To confirm that a ‘quadratic economy’ both preserves tractability and the strategic interdependency between traders, the final section outlines an example before concluding remarks are presented.

2 A Model of Bilateral Oligopoly

Bilateral oligopoly models fully strategic trade in a partial equilibrium-like environment. The economy consists of just two commodities; the first is a standard consumption commodity (and is denoted y_1) and the second is a commodity money (denoted y_2). The set of agents I is partitioned into a set of *sellers* I^S that are endowed only with the consumption good (seller i ’s endowment is denoted e_i), and a set of buyers I^B that are endowed only with the commodity money (buyer i ’s endowment is denoted m_i). It is assumed that $|I^S|, |I^B| \geq 2$ and $I^S \cap I^B = \emptyset$. Traders have preferences over the two goods that are assumed to be representable by a utility function $u_i : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ that is continuously differentiable as many times as required. $\partial_i(y)$ denotes the (absolute value of the) marginal rate of substitution of trader i at the allocation $y = (y_1, y_2)$, and p_i^* denotes her marginal rate of substitution at her endowment. Throughout, traders’ preferences are assumed to be *binormal* (i.e. both goods are normal), which implies that competitive income-expansion paths are upward-sloping.

Assumption (Binormal preferences). *The preferences of every trader $i \in I$ are such that $\partial \partial_i(y) / \partial y_1 < 0$ and $\partial \partial_i(y) / \partial y_2 \geq 0$ for all $y > 0$.*

Trade takes place via a strategic market game mechanism as follows. Each seller $i \in I^S$ decides on a portion of their endowment of the good $0 \leq x_i \leq e_i$ to supply to a ‘trading post’ to be exchanged for money. Likewise, each buyer $i \in I^B$ decides on an amount of money $0 \leq b_i \leq m_i$ to send to the trading post to be exchanged for the good. The role of the trading post is to aggregate the ‘offers’ from the sellers to $X = \sum_{i \in I^S} x_i$ and the ‘bids’ of the buyers to $B = \sum_{i \in I^B} b_i$ and to distribute the aggregate offer amongst the buyers and the aggregate bid amongst the sellers according to their respective bids and offers. The distribution rule allocates to each seller an amount of money given by their proportional share of the aggregate bid that came to the

trading post (i.e. $(x_i/X)B$), and allocates to each buyer their proportional share of the aggregate offer that came to the trading post (i.e. $(b_i/B)X$). Note that $p = B/X$ represents the price of the good denominated in units of commodity money. Note also that the trading mechanism clears the market: the proportional-sharing nature of the allocation mechanism ensures all supply is allocated amongst the buyers and all money is allocated amongst the sellers. If $B \cdot X = 0$ then the market is deemed closed, no trade takes place and traders are left with their initial endowment. The stated trading rules constitute a well-defined game and the equilibrium concept used is that of Nash equilibrium in pure strategies.

3 Bilateral Oligopoly as a ‘Dual Contest’

Autarky is always a Nash equilibrium in bilateral oligopoly.¹ An elementary but key insight in the study of *non*-autarkic equilibria in bilateral oligopoly is that the trading mechanism can be thought of as traders engaging in two linked proportional-sharing contests. In one, the sellers contest the aggregate bid sent to the market by the buyers; in the other, the buyers contest the aggregate supply sent to the market by the sellers.

Recall that in a standard Tullock contest (see, for example, Pérez-Castrillo and Verdier (1992)) each of $i = 1, \dots, n$ contestants chooses an effort level e_i in contesting a (fixed) prize R . The cost of effort for contestant i is $c_i(e_i)$ and the probability she wins the prize is given by $e_i^r / \sum_{i=1}^n e_i^r$, $r \geq 0$. In an alternative interpretation the prize can be thought of as being perfectly divisible in which case $e_i^r / \sum_{i=1}^n e_i^r$ represents the *share* of the prize awarded to contestant i . In a proportional-sharing contest $r = 1$ and the contest allocates the prize in proportion to contestants’ efforts, in which case the payoff to contestant i is given by $(e_i / \sum_{i=1}^n e_i)R - c_i(e_i)$. The *aggregative* nature of this contest (each player’s payoff depends on others’ effort only through the aggregate level of effort) can be exploited to tractably analyze Nash equilibria even with heterogeneous players (Cornes and Hartley 2005).²

¹It is readily confirmed that if the actions of all traders other than trader i are zero it is a best response for trader i to bid or offer zero depending on whether she is a buyer or a seller.

²The basic problem that can be overcome when a game is aggregative is the ‘proliferation of dimensions’ when there are many heterogeneous players in a game. To see this, the first-order condition in a proportional-sharing Tullock contest is $\frac{E_{-i}}{(e_i + E_{-i})^2} R - c'_i(e_i) \leq 0$ (with equality if $e_i > 0$, assume $c''_i \geq 0$) where $E_{-i} = \sum_{j \neq i} e_j$. Standard methods call for each contestant’s best response to be found, which is the solution in e_i to the first-order condition, denoted $\hat{e}_i(E_{-i})$. Since the domain of each contestant’s best response is different finding mutually-consistent best responses is an n -dimensional fixed point problem, which leads researchers to either concentrate on two-player games or assume homogeneity amongst players. An aggregative approach proceeds as follows. In the first-order condition replace E_{-i} with $E - e_i$. The solution in e_i to the first-order condition now gives $\tilde{e}_i(E)$, interpreted as the effort of player i consistent with a Nash equilibrium in which the aggregate effort of all players, including player i , is E (referred to as the ‘replacement function’). The important difference is that the domain of this function is the same for every player. There is a Nash equilibrium with aggregate effort E if and only if $\sum_{i=1}^n \tilde{e}_i(E) = E$: by exploiting the aggregative properties of the game a many-dimensional fixed point problem has been

The ‘trick’ in bilateral oligopoly, as first recognized by Dickson and Hartley (2008), is to separate the analysis into two ‘partial games’: in the first the aggregate bid of the buyers is fixed at $B > 0$ and the sellers play a contest in which the ‘prize’ of B is shared in proportion to their offers (the analog of effort for the sellers), i.e. a ‘revenue-sharing’ game; in the second the aggregate offer of the sellers is fixed at $X > 0$ and the buyers play a contest in which the ‘prize’ X is shared in proportion to bids (the analog of effort for the buyers), i.e. a ‘goods-sharing’ game. In each partial game with fixed prizes a Nash equilibrium is sought and the aggregative nature of each contest can be exploited to provide a tractable analysis with heterogeneous sets of buyers and sellers, just as in a standard Tullock contest.³ These ‘partial equilibria’ take into account competition *within* each side of the market and give the offers and bids of traders consistent with an equilibrium in bilateral oligopoly in which the aggregate bid and supply take the specified values. To ensure consistency *between* the sides of the market B and X must be such that the actions of sellers in contesting B aggregate to X and the bids of the buyers in contesting X aggregate to B .

Separating the analysis of bilateral oligopoly into two partial games allows the distinction between competition *within* each side of the market and *between* the sides of the market. Strategic interactions within each side of the market are transparent in the two partial games. In general, however, bilateral oligopoly cannot be analyzed as two independent contests: they are linked because there are also strategic interactions between the sides of the market, to the extent that traders on each side of the market care about the aggregate actions of traders on the opposite side (i.e. their ‘effort’ in their contest is influenced by the size of the ‘prize’ they believe they are contesting). It is precisely the strategic interaction *between* the sides of the market that makes bilateral oligopoly an interesting model.

The remainder of this section considers more carefully the partial games played by the sellers and buyers respectively, and recalls the conditions under which a non-autarkic Nash equilibrium exists.

3.1 The Sellers’ Contest

First, fix the aggregate bid of the buyers at some $B > 0$ and consider the partial game played amongst the sellers in which they contest B . The payoff to seller $i \in I^S$ in this partial game is $u_i(e_i - x, (x/x + X_{-i})B)$ and the first order condition for x to be a best response to X_{-i} is

$$\partial_i \left(e_i - x, \frac{x}{x + X_{-i}} B \right) \geq \left(1 - \frac{x}{x + X_{-i}} \right) \frac{B}{X}, \quad (1)$$

reduced to finding a fixed point of a sum of functions in two-dimensional space.

³The analogy is not perfect as the payoffs to traders in the two contests in bilateral oligopoly are not in general additively separable, as in a Tullock contest.

with equality if $x > 0$. Now define

$$\tilde{x}_i(X; B) = \left\{ x : \partial_i \left(e_i - x, \frac{x}{X} B \right) \geq \left(1 - \frac{x}{X} \right) \frac{B}{X} \right\},$$

with equality if $x > 0$. In the contest in which the size of the prize is B , $\tilde{x}_i(X; B)$ gives the offer of seller i consistent with a Nash equilibrium in which the aggregate offer of all sellers is X , called seller i 's 'replacement function'. For the aggregate offer X to be consistent with Nash equilibrium the sum of individual offers must equal this aggregate offer, i.e. X must satisfy

$$\sum_{j \in I^S} \tilde{x}_j(X; B) = X.$$

Let $\tilde{\mathcal{X}}(B)$ denote the aggregate offer that satisfies this equation, which is the aggregate offer consistent with Nash equilibrium in the sellers' contest in which the (fixed) prize is B .

Analytically, it is more tractable to consider each seller's *share* of the aggregate supply rather than the level of their supply. Let $\sigma = x/X$ and for $X > 0$ define

$$\tilde{s}_i^S(X; B) = \left\{ \sigma : \partial_i(e_i - \sigma X, \sigma B) \geq (1 - \sigma) \frac{B}{X} \right\}, \quad (2)$$

with equality if $\sigma > 0$. Then it follows that $\tilde{\mathcal{X}}(B)$ satisfies $\sum_{j \in I^S} \tilde{s}_j^S(\tilde{\mathcal{X}}(B); B) = 1$. Under the assumption of binormality certain properties of these 'share functions' are implied, which in turn give rise to desirable features of the consistent aggregate offer $\tilde{\mathcal{X}}(B)$, as the following proposition demonstrates.

Proposition 1. *Suppose the preferences of all sellers are binormal. Then for any $B > 0$ there is at most one value of $\tilde{\mathcal{X}}(B) > 0$. In addition, for $B' > B$ where it is defined, $\tilde{\mathcal{X}}(B') > (=, <) \tilde{\mathcal{X}}(B)$ if $\partial_i(y)/y_2$ is decreasing (constant, increasing) in y_2 for all $i \in I^S$.*

Proof. $\tilde{\mathcal{X}}(B)$ is the value of X where $\sum_{j \in I^S} \tilde{s}_j^S(X; B) = 1$. As such if, for fixed B , $\tilde{s}_i^S(X; B)$ is strictly decreasing in X for all $i \in I^S$ then there will be at most one value of X where the sum of share functions equals unity. Using (2), implicit differentiation of $\tilde{s}_i^S(X; B)$, after substitution of the first-order condition (1), yields

$$\frac{\partial \tilde{s}_i^S(X; B)}{\partial X} = - \frac{\frac{1}{X} \left(\partial_i(y) - y_1 \frac{\partial \partial_i(y)}{\partial y_1} \right)}{-X \frac{\partial \partial_i(y)}{\partial y_1} + B \frac{\partial \partial_i(y)}{\partial y_2} + \frac{B}{X}} < 0$$

(when preferences are binormal). This implies the aggregate share function is strictly decreasing in X . Since $\sum_{j \in I^S} \tilde{s}_j^S(\tilde{\mathcal{X}}(B); B) = 1$ if, for $B' > B$, $\sum_{j \in I^S} \tilde{s}_j^S(\tilde{\mathcal{X}}(B); B') > (=, <) \sum_{j \in I^S} \tilde{s}_j^S(\tilde{\mathcal{X}}(B); B) = 1$ it follows from the fact that the aggregate share function is decreasing in X that $\tilde{\mathcal{X}}(B') > (=, <) \tilde{\mathcal{X}}(B)$. Understanding how share functions vary with B is therefore important in understanding how the consistent aggregate offer varies with B . Implicit differentiation of $\tilde{s}_i^S(X; B)$ with respect to B , after substitution of the first-order condition and some re-arrangement, yields

$$\frac{\partial \tilde{s}_i^S(X; B)}{\partial B} = - \frac{\frac{1}{B} \left(y_2 \frac{\partial \partial_i(y)}{\partial y_2} - \partial_i(y) \right)}{-X \frac{\partial \partial_i(y)}{\partial y_1} + B \frac{\partial \partial_i(y)}{\partial y_2} + \frac{B}{X}}.$$

As such, when preferences are binormal $\partial s_i^S(X; B)/\partial B > (=, <) 0 \Leftrightarrow y_2 \partial \partial_i(y)/\partial y_2 - \partial_i(y) < (=, >) 0$, i.e. if and only if $\partial_i(y)/y_2$ is decreasing (constant, increasing) in y_2 . If this holds for all $i \in I^S$ the same is implied about the aggregate share function, giving the desired result. \square

Dickson and Hartley (2008) found the analysis of bilateral oligopoly more tractable using a change of variables to consider the behaviour of sellers consistent with a Nash equilibrium in which the price, $p = B/X$, takes a particular value. By considering market shares consistent with a Nash equilibrium in which the aggregate offer of all sellers is X and the price is p , which take the form

$$s_i^S(X; p) = \{\sigma : \partial_i(e_i - \sigma X, \sigma X p) \geq (1 - \sigma)p\} \quad (3)$$

(with equality if $\sigma > 0$), they constructed a *strategic supply function*, denoted $\mathcal{X}(p)$, that gives the aggregate offer from the sellers consistent with a Nash equilibrium in which the price is p , defined by $\sum_{j \in I^S} s_j^S(\mathcal{X}(p); p) = 1$. When preferences are binormal, several properties of strategic supply can be deduced.

Lemma 2 (Dickson and Hartley (2008), Lemma 3.2). *Suppose the preferences of all sellers are binormal. Then strategic supply is a function defined only above a cutoff price P^S , given by*

$$\sum_{j \in I^S} \max \left\{ 0, 1 - \frac{p_j^*}{P^S} \right\} = 1 \quad (4)$$

(unless $p_i^* = 0$ for all $i \in I^S$ in which case $P^S \equiv 0$), where it is positive and continuous.

3.2 The Buyers' Contest

In the buyers' contest the aggregate offer from the sellers is fixed at $X > 0$ and the buyers play a game in which X is shared in proportion to bids. The payoff to buyer $i \in I^B$ in this contest is $u_i((b/b + B_{-i})X, m_i - b)$. The first-order condition for b to be a best response to B_{-i} can be used to derive the bid of buyer i consistent with a Nash equilibrium in the partial game with aggregate offer X in which the aggregate bid of all buyers takes the value B , which is given by

$$\tilde{b}_i(B; X) = \left\{ b : \partial_i \left(\frac{b}{B} X, m_i - b \right) \leq \left(1 - \frac{b}{B} \right)^{-1} \frac{B}{X} \right\},$$

with equality if $b > 0$. Consistency of the aggregate bid B requires the sum of the individual bids to be equal to the aggregate bid, or the sum of the buyers' shares $\sigma = b/B$ to be equal to 1. For $B > 0$ define

$$\tilde{s}_i^B(B; X) = \left\{ \sigma : \partial_i(\sigma X, m_i - \sigma B) \leq (1 - \sigma)^{-1} \frac{B}{X} \right\}$$

with equality if $\sigma > 0$, then the consistent aggregate bid $\tilde{B}(X)$ is defined by $\sum_{j \in I^B} \tilde{s}_j^B(\tilde{B}(X); X) = 1$. Several properties of $\tilde{B}(X)$ can be derived from the properties of share functions when preferences are binormal.

Proposition 3. *Suppose the preferences of all buyers are binormal. Then for any $X > 0$ there is at most one value of $\tilde{B}(X) > 0$. Moreover, for $X' > X$ where it is defined, $\tilde{B}(X') > (=, <) \tilde{B}(X)$ if $y_1 \partial_i(y)$ is increasing (constant, decreasing) in y_1 for all $i \in I^B$.*

Proof. The proof of this proposition is similar to the proof of Proposition 1 for the sellers' partial game. Whilst the entire proof is omitted, the important difference is the characteristic of preferences that governs how the aggregate bid in the buyers' contest changes in response to a change in X , which results from the fact that

$$\frac{\partial \tilde{s}_i^B(B; X)}{\partial X} = \frac{\frac{1}{X} \left(y_1 \frac{\partial \partial_i(y)}{\partial y_1} + \partial_i(y) \right)}{\left| X \frac{\partial \partial_i(y)}{\partial y_1} - B \frac{\partial \partial_i(y)}{\partial y_2} - \frac{1}{(1-\sigma)^2} \frac{B}{X} \right|}$$

(when preferences are binormal). As such, the monotonicity of $\tilde{s}_i^B(B; X)$ with respect to X is governed by the sign of $y_1 \partial \partial_i(y) / \partial y_1 + \partial_i(y)$, i.e. whether $y_1 \partial_i(y)$ is increasing (constant, decreasing) in y_1 . \square

In characterizing buyers' behavior consistent with the price, it is convenient to think not of the consistent aggregate bid but the level of demand, given by the ratio of aggregate bid to price, consistent with a Nash equilibrium in which the price is p . Noting that $\mathcal{B}(p)$ is such that $\sum_{j \in I^B} s_j^B(\mathcal{B}(p); p) = 1$ where

$$s_i^B(B; p) = \{\sigma : \partial_i(\sigma B / p, m_i - \sigma B) \leq (1 - \sigma)^{-1} p\} \quad (5)$$

(with equality if $\sigma > 0$) *strategic demand* takes the form $\mathcal{D}(p) = \mathcal{B}(p)/p$.⁴ When preferences are binormal, strategic demand has several desirable properties.

Lemma 4 (Dickson and Hartley (2008), Lemmas 3.4 and 5.1). *Suppose the preferences of all buyers are binormal. Then strategic demand $\mathcal{D}(p)$ is defined only below a cutoff price P^B which is such that*

$$\sum_{j \in I^B} \max \left\{ 0, 1 - \frac{P^B}{p_j^*} \right\} = 1 \quad (6)$$

(unless $p_i^* = \infty$ for all $i \in I^B$ in which case $P^B \equiv \infty$), where it is positive, continuous and strictly decreasing in p .

3.3 Equilibrium in Bilateral Oligopoly

The analysis of the two contests played by the sellers and buyers assuming the behavior of the traders on the other side of the market is fixed considered only competition *within* each side of the market. Equilibrium in bilateral oligopoly requires not only that offers and bids constitute a Nash equilibrium in each contest, but also that traders' actions are consistent *between* the two contests. This requires, for example, B to be such that the aggregate offer in the sellers' contest $\tilde{\mathcal{X}}(B)$ consistent with this B is such that when the buyers contest this aggregate offer the aggregate bid is precisely B : $\tilde{B}(\tilde{\mathcal{X}}(B)) = B$.⁵

⁴Notice that this implies $\mathcal{D}(p)$ must satisfy $\sum_{j \in I^B} s_j^B(p \mathcal{D}(p); p) = 1$.

⁵An equivalent requirement is $\tilde{\mathcal{X}}(\tilde{B}(X)) = X$.

Bilateral oligopoly can thus be thought of as follows. Each set of traders forms a belief about the aggregate actions of the traders on the opposite side of the market and play a contest given this belief.⁶ $\tilde{\mathcal{X}}(B)$ and $\tilde{\mathcal{B}}(X)$ represent the aggregate actions of traders in the sellers' and buyers' contests, respectively, taking into account competition *within* the contests. The actions between the sides of the market must then be mutually consistent (for beliefs to be confirmed). This last requirement is as if the game is a simple two-player game where the players are the sides of the market and $\tilde{\mathcal{X}}(B)$ and $\tilde{\mathcal{B}}(X)$ represent their best responses.

In general, the properties of strategic supply and demand are more conducive to finding a non-autarkic equilibrium than utilizing the properties of $\tilde{\mathcal{X}}(B)$ and $\tilde{\mathcal{B}}(X)$. As shown in Dickson and Hartley (2008), there is a non-autarkic Nash equilibrium in bilateral oligopoly with price p if and only if strategic supply equals strategic demand: $\mathcal{X}(p) = \mathcal{D}(p)$. If strategic supply and demand do not intersect then there is no non-autarkic Nash equilibrium; the only equilibrium is autarky.

Accordingly, the properties of strategic supply and demand outlined in Lemmas 2 and 4 can be used to conclude when a non-autarkic Nash equilibrium exists.

Lemma 5 (Dickson and Hartley (2008), Theorem 4.3). *Suppose the preferences of all traders are binormal. Then there is a non-autarkic Nash equilibrium if and only if $P^S < P^B$.*

4 The Properties of Cobb-Douglas Preferences

Before deducing the implications of assuming Cobb-Douglas preferences in bilateral oligopoly this section presents the properties of such preferences. If the preferences of trader $i \in I$ take the Cobb-Douglas form then $u_i(y) = k_i y_1^{\alpha_i} y_2^{\beta_i}$ (it is often assumed that $\beta_i = 1 - \alpha_i$ with $\alpha_i < 1$). The economy is said to be a *Cobb-Douglas economy* if all traders' preferences take the Cobb-Douglas form. With Cobb-Douglas preferences $\partial_i(y) = \frac{\alpha_i}{\beta_i} \frac{y_2}{y_1}$ and the following properties are immediate.

Property 1. *For any seller $i \in I^S$ (with $e_i > 0$) p_i^* (the marginal rate of substitution at the endowment) is 0 and for any buyer $i \in I^B$ (with $m_i > 0$) $p_i^* = +\infty$. This follows because indifference curves are asymptotic to the axes of (y_1, y_2) -space: $\lim_{y_1 \rightarrow 0} \partial_i(y) = +\infty$ for any $y_2 > 0$ and $\lim_{y_2 \rightarrow 0} \partial_i(y) = 0$ for any $y_1 > 0$.*

Property 2. *Cobb-Douglas preferences satisfy binormality: $\frac{\partial \partial_i(y)}{\partial y_1} = -\frac{\alpha_i}{\beta_i} \frac{y_2}{y_1^2} < 0$ and $\frac{\partial \partial_i(y)}{\partial y_2} = \frac{\alpha_i}{\beta_i} \frac{1}{y_1} > 0$ for all $y > 0$.*

Property 3. *$\frac{\partial}{\partial y_1} \{y_1 \partial_i(y)\} = 0$. Whilst $\partial_i(y) = \frac{\alpha_i}{\beta_i} \frac{y_2}{y_1}$ is decreasing in y_1 , the product of y_1 and $\partial_i(y)$ is constant as y_1 varies. This implies that individual competitive demand from a buyer*

⁶In equilibrium all traders on a particular side of the market must share the same belief.

with Cobb-Douglas preferences is unit-elastic. To see this write the competitive first-order condition as $\partial_i([y_1 p]/p, m_i - [y_1 p]) = p$. Then implicit differentiation yields

$$\text{sgn} \left\{ \frac{d[y_1 p]}{dp} \right\} = \text{sgn} \left\{ -\frac{1}{p} \left(\partial_i(y) + y_1 \frac{\partial \partial_i(y)}{\partial y_1} \right) \right\}$$

(when preferences are binormal). Competitive demand is elastic (unit elastic, inelastic) when $\frac{d[y_1 p]}{dp} < (=, >) 0$, and since $\partial_i(y) + y_1 \frac{\partial \partial_i(y)}{\partial y_1} = \frac{\partial}{\partial y_1} \{y_1 \partial_i(y)\} = 0$ when preferences are Cobb-Douglas, demand is everywhere unit elastic.

Property 4. $\frac{\partial}{\partial y_2} \{\partial_i(y)/y_2\} = 0$. $\partial_i(y) = \frac{\alpha_i y_2}{\beta_i y_1}$ is increasing in y_2 but $\partial_i(y)/y_2$ is constant in y_2 . This implies that the individual competitive supply from a seller with Cobb-Douglas preferences is independent of the price (i.e. perfectly inelastic). To see this recall that the competitive first-order condition is $\partial_i(e_i - x, xp) = p$, implicit differentiation of which reveals

$$\text{sgn} \left\{ \frac{dx}{dp} \right\} = \text{sgn} \left\{ -\frac{1}{p} \left(y_2 \frac{\partial \partial_i(y)}{\partial y_2} - \partial_i(y) \right) \right\}$$

(when preferences are binormal). Since $y_2 \frac{\partial \partial_i(y)}{\partial y_2} - \partial_i(y) = \frac{\partial}{\partial y_2} \{y_2 \partial_i(y)\} = 0$ when preferences are Cobb-Douglas, supply is constant in p .

5 Bilateral Oligopoly in a Cobb-Douglas Economy

In the dual contest that constitutes bilateral oligopoly a key implication of assuming a Cobb-Douglas economy is that the equilibrium aggregate actions in each contest are invariant to the magnitude of the prize being contested (i.e. the aggregate actions of the traders on the opposite side of the market), as the next proposition demonstrates.

Proposition 6. In bilateral oligopoly with a Cobb-Douglas economy $\tilde{\mathcal{X}}(B') = \tilde{\mathcal{X}}(B)$ for any $B, B' > 0$ and $\tilde{\mathcal{B}}(X') = \tilde{\mathcal{B}}(X)$ for any $X, X' > 0$: the aggregate offer (bid) in the equilibrium of the sellers' (buyers') contest does not depend on the size of the prize being contested.

Proof. Proposition 1 concluded that $\tilde{\mathcal{X}}(B)$ is increasing (constant, decreasing) in B if $\frac{\partial}{\partial y_2} \{\partial_i(y)/y_2\} < (=, >) 0$. Property 4 of Cobb-Douglas preferences found that $\frac{\partial}{\partial y_2} \{\partial_i(y)/y_2\} = 0$. Thus, the consistent aggregate offer in the contest played by the sellers is independent of the aggregate bid being contested. Likewise, Proposition 3 deduced that $\tilde{\mathcal{B}}(X)$ is increasing (constant, decreasing) in X if $\frac{\partial}{\partial y_1} \{y_1 \partial_i(y)\} > (=, <) 0$, and Property 3 of Cobb-Douglas preferences found $\frac{\partial}{\partial y_1} \{y_1 \partial_i(y)\} = 0$, implying the stated result. \square

What does assuming a Cobb-Douglas economy imply about strategic supply and demand functions? It turns out that the same conditions that govern the direction of change in $\tilde{\mathcal{X}}(B)$ and $\tilde{\mathcal{B}}(X)$ when B and X change govern how $\mathcal{X}(p)$ and $\mathcal{B}(p)$ respond to p .⁷ This allows the conclusion that in a Cobb-Douglas economy strategic supply

⁷See Dickson and Hartley (2008, Lemma 5.2) for the result for strategic supply and Dickson (2012, Lemma 1) for the result for the aggregate bid.

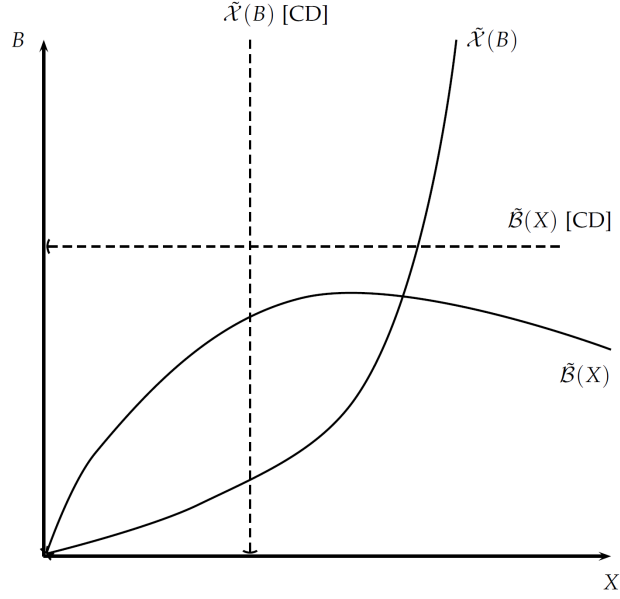


Figure 1: Typical $\tilde{\mathcal{X}}(B)$ and $\tilde{\mathcal{B}}(X)$ functions in bilateral oligopoly (solid curves), and those in a Cobb-Douglas economy (dashed lines).

is constant in p (i.e. perfectly inelastic) and, since $\mathcal{B}(p)$ (which is total revenue) is constant, that strategic demand $\mathcal{D}(p)$ is everywhere unit elastic. Thus, the same implications of Cobb-Douglas preferences for *competitive* markets (outlined in Properties 3 and 4) transfer to the strategic analogs of supply and demand in bilateral oligopoly.

Proposition 6 implies that in bilateral oligopoly with a Cobb-Douglas economy the behaviour on each side of the market is not strategically dependent on the other side of the market; the only strategic interaction occurs *within* each side of the market. As such, under the Cobb-Douglas assumption bilateral oligopoly can effectively be analyzed as two independent contests. This is not a general feature of bilateral oligopoly, as illustrated in Figure 1 which plots the consistent aggregate offer and bid in a Cobb-Douglas economy and typical $\tilde{\mathcal{X}}(B)$ and $\tilde{\mathcal{B}}(X)$ functions. The lack of strategic interdependence between the sides of the market has implications for any analysis that concerns the behaviour of one side of the market when the other side of the market changes, as in the case of a comparative static analysis or increasing the number of traders, discussed in turn in the next section.

6 The Implications of Cobb-Douglas Preferences in Bilateral Oligopoly

Bilateral oligopoly is an interesting model as it allows both buyers and sellers to behave strategically. In general, the strategic interaction is rich, being both within each side of the market and between sellers and buyers. However, as deduced in the previous section the strategic interaction between the sides of the market is absent in a Cobb-Douglas economy, which has implications for results in bilateral oligopoly.

6.1 Comparative Statics

A very interesting question to address in bilateral oligopoly concerns the effect of changes in the economic environment on equilibrium outcomes, such as the effect of an additional competitor on one side of the market. By exploiting the dual contest nature of bilateral oligopoly this question becomes tractable to address. The reason is that if the set of traders on one side of the market remains the same then the analysis of their behavior in their contest with a *fixed* prize doesn't change. In contrast, the analysis of the contest on the side of the market where the set of traders changes is affected but in a tractable way, and the 'shift' in the function representing consistent aggregate behavior with a fixed prize then implies how the equilibrium will change.

To illustrate consider a change to the set of buyers: the entry of an additional buyer; a change in buyers' endowments; or a shift in their preferences. By virtue of analyzing bilateral oligopoly as a dual contest, the consistent aggregate offer in the partial game played by the sellers in which the aggregate bid takes an arbitrary value B remains unchanged at $\tilde{\mathcal{X}}(B)$. In the partial game played by the buyers there will be a change in the aggregate bid consistent with a Nash equilibrium in which the aggregate supply takes an arbitrary value X to, say $\tilde{B}'(X)$. Using this new consistent aggregate bid function, the new equilibrium bid and offer can be found (and the effects on individual traders deduced). In general, comparative static properties are not trivial to deduce and Dickson (2012) uses strategic supply and demand functions to undertake such an analysis.

In a Cobb-Douglas economy, however, comparative statics are particularly simple: Proposition 6 concluded that the consistent aggregate offer in the partial game played by the sellers is independent of the aggregate bid of the buyers, and likewise the consistent aggregate bid of the buyers is independent of the supply of the sellers. As such, whilst in the above scenario $\tilde{B}(X)$ changes, the equilibrium value of $\tilde{\mathcal{X}}(B)$ will be unchanged because it does not depend on B . So long as the set of sellers is unchanged their equilibrium supply will be exactly the same whatever happens on the buyers' side of the market. Sellers' equilibrium *allocations* will, of course, change as the rate of exchange of the good for money will change due to the change in the equilibrium aggregate bid. Thus, if X^* and B^* were the old equilibrium aggregates, the

equilibrium supply will remain at X^* and the equilibrium aggregate bid will change to be that consistent with a Nash equilibrium in the contest played by the new set of buyers (which itself will be invariant to the equilibrium aggregate supply, if all buyers have Cobb-Douglas preferences).

In bilateral oligopoly with a Cobb-Douglas economy the lack of strategic interdependence between the sides of the market highlighted in the preceding discussion implies several interesting features may be missed. Dickson (2012) investigated the effects of entry in bilateral oligopoly, in particular looking at the effect of an increase in the number of traders on only one side of the market. Considering an increase in the number of sellers it was shown, in the context of a simple example and in a general model of bilateral oligopoly, that in contrast to the conventional wisdom in Cournot markets existing sellers may receive a higher payoff in the presence of additional sellers if the number of sellers is sufficiently few. The intuition for this result is as follows:

Buyers see an additional seller enter the market; they conjecture an increase in aggregate supply; in the contest on their side of the market they may bid more aggressively in competing for their share of the increased aggregate supply; if so, and the aggregate bid increases sufficiently, then individual sellers who increased their supply, despite receiving a smaller share of the aggregate bid, may receive a higher value of the aggregate bid; if the aggregate bid increases sufficiently then the increase in revenue to sellers will be valued more than the disutility of supplying more, leading to an increase in equilibrium payoff.

Whether this occurs depends on if, and by how much, the aggregate bid *increases* with a conjectured increase in supply. This is measured by the elasticity of strategic demand, and it is shown that a necessary condition for ‘payoff-increasing competition’ as just described is that demand is elastic, i.e. the aggregate bid increases (Dickson 2012, Corollary 7). In a Cobb-Douglas economy strategic demand is everywhere unit-elastic since the aggregate bid from the buyers is the same regardless of their beliefs about aggregate supply. Thus, when additional sellers enter a Cobb-Douglas economy the buyers may very well conjecture an increase in aggregate supply but the aggregate bid they make will not change. Consequently, existing sellers will always receive a reduction in equilibrium payoff, as the conventional wisdom suggests. The lack of strategic interdependence between the sides of the market under the Cobb-Douglas economy assumption implies interesting non-conventional results are not exhibited.

6.2 Many-Agent Limits and Convergence

In a fully strategic model of trade such as bilateral oligopoly it is interesting to investigate the sequence of equilibria as the number of traders on one or both sides of the market are replicated, in particular the relationship between the limit of this sequence and classical market models that invoke price-taking assumptions, such as Cournot and Walras equilibrium.

It is straightforward to show that if there are more traders in the contest on one side of the market then the aggregate actions consistent with a Nash equilibrium in that contest (at least weakly) increase, i.e. $\tilde{X}(B)$ increases for all B as the number of sellers increases, and $\tilde{B}(X)$ increases for all X as the number of buyers increases.⁸ Typically, an increase in the number of traders will have two effects: a *direct effect* in the contest in which a new trader is active, which serves to increase the aggregate actions of traders consistent with equilibrium in that contest; and an *indirect effect* on the opposite side of the market where traders conjecture a change in aggregate actions resulting from the presence of additional traders. As deduced in Proposition 6, in a Cobb-Douglas economy the consistent aggregate offer (resp. bid) in bilateral oligopoly does not depend on beliefs about the aggregate bid (resp. offer), and so there is no indirect effect. An implication of this is that as the number of both buyers and sellers in bilateral oligopoly with a Cobb-Douglas economy is increased (even if in an arbitrary fashion) the sequences of equilibrium aggregate supply and aggregate bid will monotonically increase, implying monotonic convergence to the Walrasian equilibrium.⁹

Bilateral oligopoly is a framework in which replication of only one side of the market can be addressed. Dickson and Hartley (2011) studied the issue of increasing the number of buyers with a fixed set of sellers, in particular investigating whether the sequence of equilibria in bilateral oligopoly converges to a Cournot equilibrium as the number of buyers increases without bound. They found that this is true only in very special cases: if and only if the elasticity of (competitive) demand at the Cournot equilibrium is unity. The intuition behind the result is as follows.

In Cournot oligopoly, sellers make output decisions taking into account the effect this has on the revenue generated in the market due to the influence of output on the market price via the (competitive) inverse demand function. In bilateral oligopoly, on the other hand, sellers make output decisions at the same time as buyers decide on their bids (the aggregation of which is revenue in the market) and as such do not consider any effect of their supply on the revenue in the market. If a seller's supply in Cournot oligopoly does influence revenue their optimal supply will be different to that in bilateral oligopoly, even when in bilateral oligopoly the buyers are many and behave as if they are price takers. The reason is that the seller's assessment of a marginal change in supply is different in the two models. If, however, demand is unit-elastic then total revenue in the market is constant and sellers in Cournot oligopoly perceive no change in revenue from a change in their supply.

⁸To see this recall that, for example, in the sellers' contest with fixed $B > 0$ the consistent aggregate supply is found as the value of X where the sum of the share functions equals one. If there is an additional share function in this sum that is positive at the value of X where the sum was previously equal to one the new sum must exceed one at this value of X . Consequently, since share functions are decreasing in X , this implies the new value of X consistent with the aggregate share function being equal to one must be greater than it was previously.

⁹Convergence of non-autarkic Nash equilibria in strategic market games to Walrasian equilibrium as the number of all traders increases is a well-established feature (see, for instance, Mas-Colell (1982)).

This implies that with unit-elastic demand the actions of sellers in bilateral oligopoly will converge towards those in Cournot oligopoly.

In a Cobb-Douglas economy individual competitive demand, and therefore aggregate competitive demand, is everywhere unit-elastic (Property 3) so the sequence of equilibria in bilateral oligopoly will always converge to the Cournot equilibrium as the number of buyers increases without bound. These deductions are consistent with the findings of Codognato (1995) who found that in a mixed exchange economy with some atoms and an atomless sector the set of ‘Cournot-Walras’ equilibria (in which the atomless sector are assumed to behave as price-takers) may be disjoint from the set of ‘Cournot-Nash’ equilibria (in which all traders engage in trade via the strategic market game mechanism without price-taking assumptions being imposed), but that they intersect when traders have Cobb-Douglas preferences (see also Busetto et. al. (2008)). Codognato and Julien (2013, Theorem 2) have recently demonstrated that the set of Cournot-Walras and Cournot-Nash equilibria coincide when the atomless sector have Cobb-Douglas preferences.

One reason for studying fully strategic trading environments is to provide a strategic foundation for models that impose price-taking assumptions. The non-convergence of the sequence of bilateral oligopoly equilibria to a Cournot equilibrium brings to the fore the fundamental question of whether the Cournot equilibrium concept does indeed have a strategic foundation. Assuming a Cobb-Douglas economy in bilateral oligopoly misses this key observation as the lack of strategic interdependence between the sides of the market implies the sequence of Nash equilibria in bilateral oligopoly happily converge to the Cournot equilibrium as the number of buyers increases without bound, which is not generally true.

6.3 Existence of Non-autarkic Equilibrium

As noted, autarky (in which no trade takes place) is always a Nash equilibrium in bilateral oligopoly. This is often seen as a trivial consequence of the trading mechanism. Whether such an autarkic equilibrium is interesting or not depends on whether it is a legitimate equilibrium in the context of the game; that is, whether it is robust to small perturbations of the game. Cordella and Gabszewicz (1998) were the first to recognize that there are economies in which autarky is a legitimate equilibrium of the game in the sense that traders could not be induced to trade even if small external bids and offers are put on the market. They coined autarky ‘nice’ under such circumstances. This issue was further investigated by Busetto and Codognato (2006) who considered whether autarky remained an equilibrium if *any* (not just small) external bids and offers are made to the market (in the same proportion), and coined it ‘very nice’ if so. Dickson and Hartley (2012) showed that autarky is nice if and only if it is the only equilibrium in the game, and very nice if and only if no gains from trade exist, and proceeded to consider whether there are economies in which autarky is nice but not

very nice; that is, economies in which there are gains from trade but no trade takes place. This is an important question in bilateral oligopoly: as in Cournot competition the volume of trade is expected to be lower than a comparison with a competitive equilibrium, but does the fact that there are strategic traders on *both* sides of the market mean that trade may fail to take place at all?

Recall from Lemma 5 that a non-autarkic Nash equilibrium exists in bilateral oligopoly if and only if $P^S < P^B$. In bilateral oligopoly gains from trade exist if $\min_{j \in I^S} \{p_j^*\} < \max_{j \in I^B} \{p_j^*\}$, so the question is whether this condition implies $P^S < P^B$ (i.e. the existence of gains from trade implies trade will take place). Dickson and Hartley (2012, Proposition 8) demonstrated that that, in general, it does not: $P^S \geq \min_{j \in I^S} \{p_j^*\}$ and $P^B \leq \max_{j \in I^B} \{p_j^*\}$ and so whenever these inequalities are strict it is possible that even if gains from trade exist the economy could exhibit $P^S \geq P^B$ and so the only Nash equilibrium is autarky. Rather, in order for there to be a non-autarkic Nash equilibrium there need to be ‘sufficient’ gains from trade which may, for example, require a sufficiently large number of traders.

Can this important feature be exhibited in a Cobb-Douglas economy? The answer is no. The reason can be deduced from Property 1 of Cobb-Douglas preferences: for $i \in I^S$ $p_i^* = 0$ (recall that p_i^* is the marginal rate of substitution at the endowment) which implies by definition that $P^S = 0$ (defined in (4)); and for $i \in I^B$ $p_i^* = +\infty$ which implies by definition that $P^B = +\infty$ (defined in (6)). Thus, in a Cobb-Douglas economy there will always be gains from trade and there will also always exist a non-autarkic Nash equilibrium in bilateral oligopoly regardless of the number of traders (so long as there are at least two buyers and sellers).

A Shapley equilibrium (Shapley 1976) exists if either there is an equilibrium in which trade takes place, or autarky is very nice (see Busetto and Codognato (2006) or Dickson and Hartley (2012)). In a Cobb-Douglas economy the existence of a Shapley equilibrium is guaranteed. However, this conclusion is misleading as it is not generally true: in more general bilateral oligopoly environments there are economies in which autarky is nice but not very nice.

7 An Alternative Example

The appeal of assuming the preferences of traders in bilateral oligopoly are Cobb-Douglas lies in its tractability. As has been shown, however, making this assumption has several known pitfalls. The danger in making the assumption to undertake exploration of new ideas in strategic market games is that interesting phenomena may not be identified due to the lack of strategic interdependency between the sides of the market. An alternative example that retains most of the tractability of Cobb-Douglas preferences is that of *quadratic preferences*. Whilst such preferences are quasi-linear in the commodity money and so imply zero income effects, and some consideration of appropriate parameter restrictions is required to ensure examples work nicely, they

provide a much richer environment in which to study strategic interactions in markets.¹⁰

To illustrate, suppose that the preferences of each trader $i \in I$ in bilateral oligopoly are given by $u_i(y) = \alpha_i y_1 - \frac{\gamma_i}{2} y_1^2 + y_2$ where it is assumed that $\alpha_i > \gamma_i > 0$, in which case $\partial_i(y) = \alpha_i - \gamma_i y_1$. Further, assume all traders have the same preference parameters¹¹ ($\alpha_i = \alpha$ and $\gamma_i = \gamma$ for all $i \in I$) and that there are k buyers each of which has a unit endowment of money and n sellers each of which has a unit endowment of the good. With this specification $p_i^* = \alpha - \gamma$ for all $i \in I^S$ and $p_i^* = \alpha$ for all $i \in I^B$ so gains from trade exist.

Looking first at the sellers to construct strategic supply, (3) can be used to deduce that, so long as it is positive, each seller's share function takes the form $s^S(X; p) = \frac{p - (\alpha - \gamma)}{\gamma X + p}$. The solution in X to $ns^S(X; p) = 1$ finds strategic supply when the price is p which reveals that $\mathcal{X}(p) = -\frac{n(\alpha - \gamma)}{\gamma} + \frac{n-1}{\gamma} p$, with P^S , the price below which strategic supply is not defined, being given by $P^S = \frac{n}{n-1}(\alpha - \gamma)$ (which can be deduced from (4)).

To construct strategic demand for the buyers, (5) implies that, so long as it is positive, each buyer's share function takes the form $s^B(B; p) = \frac{\alpha p + \gamma B - \sqrt{(\alpha p + \gamma B)^2 - 4\gamma B p(\alpha - p)}}{2\gamma B}$. The solution in B to $ks^B(B; p) = 1$ gives the consistent aggregate bid when the price is p , which takes the form $\mathcal{B}(p) = \frac{k\alpha}{\gamma} p - \frac{k^2}{(k-1)\gamma} p^2$. As such, strategic demand ($\mathcal{B}(p)/p$) is given by $\mathcal{D}(p) = \frac{k\alpha}{\gamma} - \frac{k^2}{(k-1)\gamma} p$ and P^B , the value of p above which strategic demand is undefined, is $P^B = \frac{k-1}{k}\alpha$.

Note that, as implied by quadratic preferences for *competitive* supply and demand functions, strategic supply and demand functions are linear. Note also that in a 'quadratic economy', despite the fact that there are gains from trade, strategic supply and demand may not cross so there may be no non-autarkic Nash equilibrium: even though gains from trade exist it could be possible that $P^S = \frac{n}{n-1}(\alpha - \gamma) \geq \frac{k-1}{k}\alpha = P^B$ (for example when $n = k = 2$, $\alpha = 2$, $\gamma = 3/2$ $P^S = P^B = 1$). If they do cross then the price at the Nash equilibrium can be found by equating the supply and demand functions and solving for p , after which the equilibrium aggregate supply and bid can be found, which in turn allows individual offers and bids to be deduced from the equilibrium values of share functions.

To demonstrate that a quadratic economy does indeed capture the richness of the strategic interaction in bilateral oligopoly as claimed, consider the aggregate actions of traders consistent with equilibrium in their respective contests. In a Cobb-Douglas economy these did not depend on the actions of the traders on the opposite side of the

¹⁰Groh (1999) used quadratic preferences fruitfully to study the difference between sequential- and simultaneous-moves in bilateral oligopoly.

¹¹Allowing buyers to have different preferences to sellers merely adds to the notation. The method also allows for traders on each side of the market to be heterogeneous, but with some added complexity in the calculations.

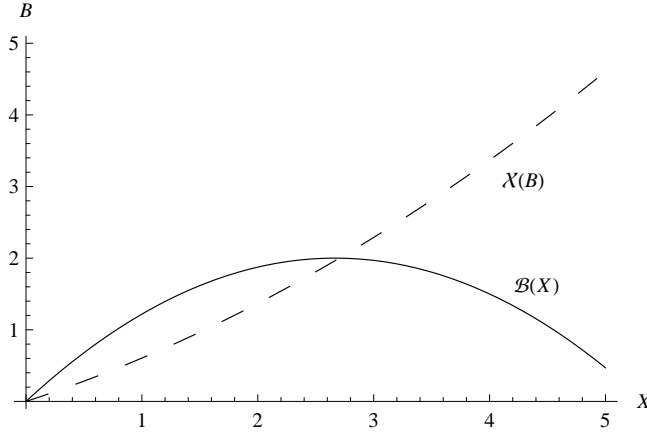


Figure 2: Consistent aggregate bid and offer functions in a quadratic economy where $n = 10$; $k = 4$; $\alpha = 2$; and $\gamma = 3/2$.

market. In a quadratic economy each seller's share function expressed as a function of X and B takes the form $\hat{s}^S(X; B) = \frac{B - X(\alpha - \gamma)}{B + \gamma X^2}$ (so long as it is positive). The solution in X to $n\hat{s}^S(X; B) = 1$ gives the consistent aggregate supply in the contest in which the aggregate bid is B , which takes the form $\hat{X}(B) = -n\frac{\alpha - \gamma}{2\gamma} + \frac{1}{2\gamma} \sqrt{n^2(\alpha - \gamma)^2 + 4\gamma(n - 1)B}$. This clearly varies with B : if the sellers conjecture a higher aggregate bid then the aggregate offer from contesting this increased revenue will increase.

To find the consistent aggregate bid in the buyers' contest note that, expressed as a function of the aggregate bid and aggregate offer, a buyer's share function takes the form $\hat{s}^B(B; X) = \frac{(\alpha + \gamma X) - \sqrt{(\alpha + \gamma X)^2 - 4\gamma(\alpha X - B)}}{2\gamma X}$ (so long as it is positive). The consistent aggregate bid in a contest in which the supply is X is the solution in B to $k\hat{s}^B(B; X) = 1$ which takes the form $\hat{B}(X) = X \left(\frac{k-1}{k}\alpha - \frac{k-1}{k^2}\gamma X \right)$. Again the aggregate bid in the buyers' contest is dependent on the aggregate supply from the sellers, in contrast to the Cobb-Douglas case. In the buyers' contest the consistent aggregate bid may change non-monotonically as the buyers' conjectured aggregate supply changes: increases in the conjectured aggregate supply when it is small will always give rise to a higher aggregate bid but, depending on the parameters, further increases beyond a particular level may imply a reduction in the consistent aggregate bid.

For an economy in which there are 10 sellers and 4 buyers and $\alpha = 2$, $\gamma = 3/2$ the consistent aggregate bid and supply functions are plotted in Figure 2. As is evident, a 'quadratic economy' provides a much richer environment in which to analyze the properties of bilateral oligopoly equilibria since it does not assume away the strategic

interaction between the sides of the market. Whereas, for example, in a Cobb-Douglas economy an increase in the number of sellers (which shifts the consistent aggregate supply function to the right) has the effect to simply increase the equilibrium aggregate supply without having an effect on the equilibrium aggregate bid, in a quadratic economy there will be an effect on the equilibrium aggregate bid, the qualitative nature of which depends on the number of traders and their preferences.

8 Concluding Remarks

Bilateral oligopoly is a simple model that allows buyers as well as sellers to behave strategically in exchanging a good via a strategic market game mechanism. It can be viewed as two linked contests: in one the sellers contest the aggregate bid of the buyers; in the other the buyers contest the aggregate offer of the sellers. Fixing the prize in each contest, the behaviour of the traders on each side of the market consistent with equilibrium in which the aggregate actions of the traders on the opposite side of the market are fixed can be found, taking into account competition *within* that side of the market. Equilibrium in bilateral oligopoly is found when the behavior of traders on each side of the market is consistent with each other, which takes into account competition *between* the sides of the market. It is this richness of strategic interaction that makes bilateral oligopoly interesting.

The unfortunate consequence of assuming a Cobb-Douglas economy in this framework is that there is no strategic interdependence between the sides of the market: the consistent actions of traders on each side of the market are the same regardless of the actions of traders on the opposite side of the market. Making this assumption thus misses a key feature of trade within bilateral oligopoly and has known implications when analyzing comparative static properties of equilibrium and when investigating many-trader limits. Caution is expressed when exploring new ideas in fully strategic markets: investigation using a Cobb-Douglas example may very well miss interesting phenomena. Rather, alternative assumptions should be pursued that preserve the key feature of strategic dependence between the sides of the market; quadratic preferences preserve most of the tractability of Cobb-Douglas preferences but also preserve the richness of the strategic interaction.

References

- Busetto, F. and G. Codognato; 2006; ““Very nice” trivial equilibria in strategic market games”; *Journal of economic theory*; 131, pp 295-301.
- Busetto, F., G. Codognato and S. Ghosal; 2008; “Cournot-Walras equilibrium as a sub-game perfect equilibrium”; *International journal of game theory*; 37, pp 371-86.

- Codognato, G.; 1995; "Cournot-Walras and Cournot Equilibria in Mixed Markets: A Comparison"; *Economic theory*; 5, pp 361-70.
- Codognato, G. and L. Julien; 2013; "Noncooperative oligopoly in markets with a Cobb-Douglas continuum of traders"; *mimeo*.
- Cordella, T. and J. Gabszewicz; 1998; "'Nice' trivial equilibria in strategic market games"; *Games and economic behavior*; 22, pp 162-9.
- Cornes, R. and R. Hartley; 2005; "Asymmetric contests with general technologies"; *Economic theory*; 26, pp 923-46.
- Dickson, A.; 2012; "The effects of entry in bilateral oligopoly"; *mimeo*.
- Dickson, A. and R. Hartley; 2008; "The strategic Marshallian cross"; *Games and economic behavior*; 64, pp 514-32.
- Dickson, A. and R. Hartley; 2011; "Bilateral oligopoly and quantity competition"; *Economic theory*; forthcoming, available online October 2011, DOI: 10.1007/s00199-011-0676-9.
- Dickson, A. and R. Hartley; 2012; "On 'nice' and 'very nice' autarkic equilibria in strategic market games"; *The Manchester school*; forthcoming, available online August 2012, DOI: 10.1111/j.1467-9957.2012.02318.x.
- Gabszewicz, J.J. and P. Michel; 1997; "Oligopoly equilibrium in exchange economies"; in: Eaton, B. and R. Harris (Eds.), *Trade, technology and economics: essays in honor of Richard G. Lipsey*; Elgar, Cheltenham, UK; pp. 217-40.
- Giraud, G.; 2003; "Strategic market games: an introduction"; *Journal of mathematical economics*; 39 pp 355-75.
- Groh, C.; 1999; "Sequential moves and comparative statics in strategic market games"; *mimeo*.
- Mas-Colell, A.; 1982; "The Cournotian foundations of Walrasian equilibrium theory: an exposition of recent theory"; in *Advances in Economic Theory*, Hildenbrand, W (ed.); Cambridge University Press; pp 183-224.
- Pérez-Castrillo, D. and T. Verdier; 1992; "A general analysis of rent-seeking games"; *Public choice*; 73, pp 335-50.
- Shapley, L.; 1976; "Non-cooperative general exchange"; in S. Lin (ed), *Theory of measurement of economic externalities*; Academic press.
- Shapley, L. and M. Shubik; 1977; "Trade using one commodity as a means of payment"; *Journal of political economy*; 85, pp 937-68.